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Optimal pricing and capacity choice for a public service under risk of interruption

Fred SCHROYEN [1] and Adekola OYENUGA[2]

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Abstract

We develop rules for pricing and capacity choice for an interruptible service that recognise the interdependence between consumers' perceptions of system reliability and their market behaviour. Consumers post *ex ante* demands, based on their expectations on aggregate demand. Posted demands are met if *ex post* supply capacity is sufficient. However, if supply is inadequate all *ex ante* demands are proportionally interrupted. Consumers' expectations of aggregate demand are assumed to be rational. Under reasonable values for the consumer's degrees of relative risk aversion and prudence, demand is decreasing in supply reliability. We derive operational expressions for the optimal pricing rule and the capacity expansion rule. We show that the optimal price under uncertainty consists of the optimal price under certainty plus a markup that positively depends on the degrees of relative risk aversion, relative prudence and system reliability. We also show that any reliability enhancing investment - though lowering the operating surplus of the public utility - is socially desirable as long as it covers the cost of investment.

Keywords: service interruption, rationing, system reliability, second-best pricing, capacity choice, prudence.

JEL Classification: D11, D24, D45, H42, Q25

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1 Introduction

1.1 Motivation

The San Francisco Public Utilities Commission (SFPUC) provides water to the citizens of San Francisco and–under contractual agreement with 29 wholesale water agencies–to 1.6 million additional customers within three Bay Area counties. Overall, 2.4 million people receive 260 million gallons (over 1 billion liter) of water per day. Most of the water is imported from the Sierra Nevada, delivered through the Hetch Hetchy aquaducts. The most serious threat to the water supply of the Bay Area is a drought, through its impact on the Sierra Mountains' snowpack which feeds the Hetch Hetchy water supply system. How should the SFPUC, or any other public utility providing a service with uncertain capacity, set its price? And how should it evaluate the desirability of its infrastructure investments? Our advice is: (i) calculate the price according to a standard inverse elasticity rule, corrected for the marginal cost of public funds (eq (4.8)) and add to it a mark-up that depends positively on the degrees of relative risk aversion and prudence, and negatively on the degree of system reliability (eq (4.9)); and (ii) expand capacity if the ensuing total reliability improvement, when multiplied with the degree of relative risk aversion and the marginal cost of production, exceeds the cost of investment (eq (4.14)).

In general, the supply of services by certain public utilities–electricity, gas, water–is characterized by an inherent uncertainty: power generating capacities are subject to temporary failures, the variability of surface water levels are not entirely predictable, and likewise for the inflow to water and gas reservoirs.[1] In addition, the quality of the delivered service may not be fully controllable (risk of pollution). Insulating consumers of these services for any supply risk, that is, guaranteeing a 100% service reliability, would in most cases require investments in infrastructure that are prohibitively costly. The first best solution is a set of contingency contracts–clear agreements for the delivery of units of service under well defined contingencies, which are paid in advance. By buying rights to such contingency deliveries, a consumer can–should he wish–secure himself a certain delivery in the future. Although new markets have been developed in recent years to insure against variations in temperature, precipitation, and events like drought, fall freeze, etc., such markets remain often closed to the general public. An alternative is the let consumers face spot prices that balance demand and supply at any

[1] Similarly, demand for these services displays a variability that may only be described statistically. Often, factors that cause a drop in supply capacity, such as a drought, will also cause demand to rise. In this paper, we abstract from any demand variability.

time. There are two problems with such an arrangement. First, it may be very costly to inform consumers in real time about the governing price level. Second, it may impose on consumers a considerable price risk. The price elasticity for residential water demand in the Bay Area is $-.176$ (SFPUC, 2007: 21). A 10% supply reduction would require prices to rise by 57%.[2] For the Gironde area in France, Nauges and Reynaud (2001) estimate the short run demand elasticity for domestic water use at $-.08$. With such an inelastic demand, a 10% supply reduction would require a price increase of 125%![3] Politicians are reluctant to allow the service price of public utilities swing that much. Irrespective of whether price stability is an objective or a constraint, the balancing of a variable supply with demand calls for quantity rationing.

In this paper, we are interested in the optimal policies for pricing and capacity choice of an uncertain supply under two conditions: (i) that prices are be kept stable, and (ii) that in the event of excess demand, the service is rationed in proportion to notional demands–the demands that would ensue at the (stable) price. First, proportional rationing is frequently practiced. In the case of the Bay Area water supply, the master contract between the SFPUC and its wholesale customers explicitly stipulates a proportional rationing rule (SFPUC, 2007: 17). But even when a proportional rationing rule is not literally practised, it is not uncommon that 'good' or 'loyal' customers are being prioritized by the service supplier in case of excess demand. The rationing rule is then said to be manipulable (in the sense of Benassy, 1977: 152) because customers can influence their share of the scarce supply by signalling a larger demand. A proportional rationing rule is the prime example of a manipulable allocation rule, and its study is useful to get insight in other situations where customers can exert influence on the amounts of the service finally allocated to them. Though our focus on a proportional rationing rule is motivated on these positive grounds, we note that there exist normative reasons for such a rule (see Moulin, 2000).

1.2 Relation to the literature

Our paper is closely related to the literature on peak-load pricing under demand and supply uncertainty. The main presumption of this literature is

[2] Ignoring scale effects on the willingness to pay.

[3] Using a data set covering 1142 large industrial and commercial customers in Northern California, Borenstein (2007) calculated customer bills under time-invariant, time-of-use, and real-time pricing (RTP) schemes and found that, after adjusting for seasonal variation, the coefficient of variation of a customer's bill is on average nearly five time larger under RTP than under the time-of-use structure that they typically face.

that both spot pricing and contingency contracts are allocation mechanisms that are too costly to implement and therefore that recourse has to be taken to other rationing mechanisms to bring demand in line with available supply. For example, Brown and Johnson (1969), Turvey (1970), Visscher (1973), Meyer (1975), Carlton (1977); and Crew and Kleindorfer (1978) have focused on demand uncertainty while Chao (1983), Fakhraei, Narayanan, Hughes (1984), Coate and Panzar (1989), and Kleindorfer and Fernando (1993) have extended the analysis to include supply uncertainty. Typical for this part of the literature is that rationing takes place on the basis of characteristics that the are assumed to be observable to the provider, such as outage costs and/or willingness to pay. Parallel to it, a literature has developed where rationing takes place on the basis of unobservable but revealed characteristics of consumers. Examples are self rationing (Panzar and Sibley, 1978, Woo, 1990, Doucet and Roland, 1993) and priority servicing (Marchand, 1974, Chao and Wilson, 1987, and Wilson, 1989a,b). For a detailed survey of both strands of literature, see Crew, Fernando and Kleindorfer (1995).[4]

In this literature, there are very few papers that explicitly model how consumers formulate their demand based on the perceived reliability of that service. One exception is the paper by Coate and Panzar (1989) on random service rationing.[5] Inspired by Rees (1980), they let risk neutral firms decide on their capital equipment before knowing whether electricity will be available for production. In doing so, firms assign a probability to being blacked out (the complement of the expected system reliability). Once capital is installed, demand for electricity follows from short run profit maximization. *Ex post*, if actual electricity supply falls short of aggregate demand, a fraction of firms is blacked out in a random way. The model is then closed by requiring that the mathematical expectation of the actual black out probability equals the anticipated black out probability (rational expectations). The authors show that electricity demand positively depends on service reliability and characterise the optimal pricing and capacity choice for the public utility.

[4] The literature often focusses on consumers and producers. A recent literature discusses the role of interruptible service contracts on deregulated power markets as instruments for hedging the wholesale market exposure of retail suppliers to a volatile spot price; see Gerda and Varaiya (1993), Kamat and Oren (2002), Baldick et al. (2006), and Rocha and Siddiqui (2008).

[5] In his model on public utility pricing with uncertain demand, Tschirhart (1980) adds system reliability as an explanatory variable to the mean demand equation, and assumes that higher reliability leads to higher mean demand. He does not derive the demand schedule on the basis of perceived reliability.

1.3 The role of the degree of relative prudence

In the present paper, we study the optimal pricing and capacity choice by a public utility when inadequate supply is allocated across risk averse consumers according to a proportional rationing rule. Consumers post *ex ante* demands for a designated consumption period. These demands will be met if *ex post* supply capacity is sufficient. If not, all *ex ante* demands will be proportionally interrupted. The anticipated system reliability is determined by anticipated aggregate demand. As Coate and Panzar, we assume rational expectations: anticipated demand coincides with aggregate *ex ante* demand. Unlike with random rationing, however, the proportional rationing rule is manipulable in that a single consumer may influence his *personal* service reliability by over/understating his demand. When risk-averse consumers can influence their own service reliability, how they respond to system reliability is an important factor in defining optimal price and capacity levels. In this respect, the concept of precautionary behaviour plays a crucial role. Such behaviour follows from a positive third derivative of the consumer's concave utility function, a property coined prudence by Kimball (1990).

The original discussion of precautionary behaviour is in terms of a consumer's savings decision when facing increased uncertainty with respect to future labour income or the return to savings–see Leland (1968), Sandmo (1970) and Kimball (1990). This discussion was recently closed by Eeckhoudt and Schlesinger (2008). In particular, they show that whenever relative risk aversion exceeds 1, consumers will save more when the rate of return distribution undergoes a first degree stochastically dominating shift (i.e., lower rates of return become more likely); and that whenever relative prudence exceeds 2, consumers will save more when the rate of return distribution undergoes a 2nd degree increase in risk (a.k.a. a mean preserving spread).

We show that these conditions on relative risk aversion and relative prudence will also govern the consumer's responses to increased uncertainty in our model. Sufficiently risk averse and prudent consumers will react to increased uncertainty by expanding *ex ante* demand. In fact, whenever the perceived reliability of the system goes down, *ex ante* demand goes up–exactly the opposite reaction as in Coate and Panzar's (1989) model. This result underscores the interdependence of system reliability, rationing rule and demand behaviour. We show that the optimal price is one that internalises the external effect of individual demands on system reliability. We also show that although infrastructure investments which increase the reliability of supply will reduce producer operating surplus, they typical produce a more than compensating increase in consumer surplus, a result–we believe–that calls for regulation of infrastructure decisions.

2 The model

A public utility is the sole supplier of a good or service. The cost of production and delivery is a constant c per unit. There is a continuum of consumers with mass normalised to 1 that all have a quasi-linear utility function over the consumption of the service, w, and a numéraire commodity, Y: $U(w,Y) = u(w) + Y$, where it is assumed that $u' > 0$, $u'' < 0$, $u''' > 0$ and $\lim_{w \to 0} u'(w) = +\infty$. For future reference, we define the coefficients of relative risk aversion and relative prudence with respect to the service as $R_r(w) \stackrel{\text{def}}{=} -\frac{u''(w)w}{u'(w)}$ and $P_r(w) \stackrel{\text{def}}{=} -\frac{u'''(w)w}{u''(w)}$, respectively.

Two remarks are in place. First, note that with constant relative risk aversion (CRRA), $P_r \equiv R_r + 1$.[6] Second, the above utility function implies that the price elasticity for the w-good equals $\frac{1}{R_r(w)}$. Table 1 displays typical estimates for short run demand elasticities for gas, electricity and water. The low estimates imply values for R_r well exceeding 2. This is useful to keep in mind when evaluating the results later on in the paper.

Table 1. Demand elasticity estimates for selected utilities.

	residential	commercial	industrial
electricity[a]	.16	.28	.39
natural gas[a]	.15	.28	.26
water	.08[b]		.29[c]

[a] Lin *et al.* (1987, p 250): United States
[b] Nauges and Reynaud (2001, p 181): Gironde (France)
[c] Reynaud (2003, p 227) Gironde (France)

Each consumer has an exogenous income at his disposal. In our model, consumers may be heterogeneous in terms of income as long as everybody's income is high enough. To simplify notation, however, we assume that incomes are identical and equal to m. If p denotes the price per unit, the representative consumer has $Y = m - pw$ left for consumption of the numéraire.

[6] Pivotal values for R_r and P_r are 1 and 2, respectively. Eeckhoudt et al. (2009) show how using simple gambles one can elicit whether or not a respondent's degree of relative risk aversion and prudence exceeds these pivotal values.

Total supply is represented by a random variable T with a commonly known cumulative distribution function $F(T)$. The realisation of this variable is exogenous to the consumer. Supply is thus uncertain and the extent to which aggregate demand X^a exceeds realised supply is the level of supply inadequacy or excess demand.

The consumer's perception of supply being adequate is given by $\Pr(T > X^e) = 1 - F(X^e)$, where X^e is the consumer's expectation regarding the aggregate demand. The assumed rationality of this expectation requires that $X^e = X^a$.

It is commonly known that a positive level of excess demand will result in consumption being interrupted or rationed off. In general, the realised consumption of the service by person i, w_i, may be written as a function of the *ex ante* demands by all agents, (x_i, x_{-i}), as well as the available capacity T, $w_i = f_i(x_i, x_{-i}, T)$. We assume that the rationing functions $f_i(\cdot)$ follow what is called a proportional rationing rule, i.e.,

$$w_i = x_i \cdot \min\{1, \frac{T}{\int_0^1 x_i \mathrm{d}i}\}.$$

The scheme thus unfolds as follows. (1) The utility announces the price p in advance of the period. (2) Consumers choose their *ex ante* demand x. (3a) If the realised supply is adequate, then x will default as uninterrupted consumption. (3b) If the realised supply is inadequate, consumption is curtailed to $\frac{T}{X^a}x$. (4) Consumers pay for the delivered portion of x at the announced price p, and consumption takes place.

3 Consumer behaviour

The first question we answer is how the consumer behaves in choosing his *ex ante* demand and to what extent this demand would be influenced by the prospect of him being interrupted.

The consumer's *ex ante* demand is the solution to the following utility maximisation problem:

$$\begin{aligned} \max_x V = & \int_0^{X^e} \left[u\left(\frac{T}{X^e}x\right) + m - p\frac{T}{X^e}x \right] \mathrm{d}F(T) \\ & + \left[u(x) + m - px\right]\left[1 - F(X^e)\right]. \end{aligned} \tag{3.1}$$

If a situation with inadequate supply is expected with some positive probability, then $\Pr(T \geq X^e) < 1$ and $F(X^e) > 0$. The demand $\hat{x}$ that solves problem (3.1) must satisfy the necessary condition:

$$u'\left(\hat{x}\right)\left[1-F\left(X^{e}\right)\right]+\int_{0}^{X^{e}}u'\left(\frac{T}{X^{e}}\hat{x}\right)\frac{T}{X^{e}}\mathrm{d}F\left(T\right)=p\cdot r_{F}(X^{e}), \tag{3.2}$$

where $r_F(X^e)$ is defined as the consumer's perception of the degree of system reliability:

$$r_{F}(X^{e})\stackrel{\text{def}}{=}F\left(X^{e}\right)E\left[\frac{T}{X^{e}}\mid T\leq X^{e}\right]+\left[1-F\left(X^{e}\right)\right]; \tag{3.3}$$

with probability $F(X^e)$ there will be a supply shortage and only a fraction $E\left[\frac{T}{X^e}\mid T\leq X^e\right]$ of total demand is expected to be satisfied, while with probability $1-F(X^e)$ no shortage occurs and demand is entirely met. Useful properties of $r_F(\cdot)$ are:

$$r_{F}(X^{e}) = 1-\frac{1}{X^{e}}\int_{0}^{X^{e}}F\left(T\right)\mathrm{d}T, \tag{3.4a}$$

$$r_{F}'(X^{e})X^{e} = -F\left(X^{e}\right)E\left[\frac{T}{X^{e}}\mid T\leq X^{e}\right]<0, \text{ and} \tag{3.4b}$$

$$\lim_{X^{e}\rightarrow 0}r_{F}(X^{e}) = 1. \tag{3.4c}$$

Applying integration by parts on the left-hand side of (3.2) then allows us to rewrite this first-order condition as (see appendix):

$$u'\left(\hat{x}\right)+\int_{0}^{X^{e}}\frac{u'\left(\frac{T}{X^{e}}\hat{x}\right)\left[R_{r}\left(\frac{T}{X^{e}}\hat{x}\right)-1\right]}{X^{e}}F\left(T\right)\mathrm{d}T=p\cdot r_{F}(X^{e}). \tag{3.5}$$

We now compare this first order condition with the case where supply is expected to be adequate in the sense that $F(T)=0$ for all $T\leq X^e$. Then $r_F(X^e)=1$, and the optimal order x^* must satisfy

$$u'\left(x^{*}\right)=p. \tag{3.6}$$

Comparing (3.5) with (3.6), shows that there are two reasons for ordering more under inadequate supply. First, the expected price $p\cdot r_F(X^e)$ lies below the nominal price p. Second, the left-hand side of (3.5) contains a term that is not present in (3.6). We call this term the *marginal risk premium effect.* It accommodates for the utility consequences of a marginal ordered unit in those states of the world where supply is insufficient. Because the consumer faces a multiplicative rather than additive risk, risk aversion alone is not sufficient for a positive risk premium. Only if relative risk aversion exceeds unity will a marginal order provide a hedge against the consumption risk. This is the second reason for posting a higher demand than under certainty. We write demand as $\hat{x}=\hat{x}\left(p,X^{e},F(\cdot)\right)$.

Proposition 1 *If inadequate supply is expected with some positive probability, and $R_r > 1$, the consumer will post a larger ex ante demand than when supply is deemed adequate.*

The second-order condition may be written as (see appendix):

$$SOC_{\hat{x}} = -\frac{\widehat{u}'}{\widehat{x}} \left\{ \widehat{R}_r + \frac{1}{X^e} \int_0^{X^e} \frac{\widetilde{u}'}{\widehat{u}'} \widetilde{R}_r (\widetilde{P}_r - 2) F(T) \mathrm{d}T \right\} < 0, \tag{3.7}$$

where a $\widehat{}$ above an expression means evaluation at $\widehat{x}$, while a $\widetilde{}$ means evaluation at $\frac{\widehat{x}T}{X^e}$. A sufficiently high relative prudence thus ensures that the second-order condition is verified.

3.1 Comparative statics at the individual level

In this section, we investigate how the consumer who expects interruptions adjusts his *ex ante* order because of marginal changes in p, and X^e as well as marginal changes in the uncertainty surrounding the supply capacity.

Simple comparative statics on (3.5) show that:

$$\frac{\partial \hat{x}}{\partial p}(-SOC_{\hat{x}}) = -r_F(X^e) < 0. \tag{3.8}$$

It is easy to show that with CRRA preferences, the price elasticity is $-\frac{1}{R_r}$.

A marginal price increase will reduce maximal expected utility with the expected delivered amount:

$$\frac{\partial V}{\partial p} = -\widehat{x}\ r_F(X^e) < 0. \tag{3.9}$$

Proposition 2 *The consumer's ex-ante demand is decreasing in the price p while a marginal price increase reduces expected utility with the reliable part of the posted demand.*

How will the consumer's demand respond to a small change in the expected aggregate demand? Again taking comparative statics on (3.5) gives:

$$\begin{aligned} \frac{\partial \hat{x}}{\partial X^e}(-SOC_{\hat{x}}) \;=\; & \frac{1}{X^e} \left\{ \int_0^{X^e} \widetilde{u}' \left[\left(\widetilde{R}_r - 1 \right)^2 - \widetilde{R}_r' T \right] \frac{F(T)}{X^e} \mathrm{d}T \right. \\ & \left. + \widehat{u}' \left(\widehat{R}_r - 1 \right) F(X^e) - r_F'(X^e) X^e p \right\} . \end{aligned} \tag{3.10}$$

The last term on the right-hand side captures the effect on the marginal expected outlay of an order. Since an increase in expected demand reduces the reliability rate (cf (3.4b)), so does the marginal expected outlay, and this encourages a higher *ex ante* order. The first two terms in curly brackets account for the effect on the marginal risk premium. Two sets of sufficient conditions are identifiable for this effect to be positive. The first is that relative risk aversion is larger than 1 but falling (the latter being equivalent to $1 + R_r < P_r$). The second is revealed by noting that the term in square brackets may be rewritten as $1 + \left(\widetilde{P}_r - 3\right) \widetilde{R}_r$ (see appendix). Therefore, a relative prudence larger than 3 and a relative risk aversion exceeding 1 again ensure that $\frac{\partial \widehat{x}}{\partial X^e} > 0$. With the range for R_r mentioned earlier, and $P_r \simeq R_r + 1$, these conditions will be verified.

The effect of a small increase in anticipated aggregate demand on maximal expected utility is

$$\frac{\partial V}{\partial X^e} = \frac{\widehat{x}}{X^e} \left[u'(\widehat{x}) - p\right] (1 - F(X^e). \tag{3.11}$$

Since $u'(x^*) = p$, this effect is negative under the same conditions that give $\widehat{x} > x^*$. This suggests that there is a negative demand externality: higher expectations about aggregate demand boost individual demand and reduce individual welfare. This externality will play an important role in the optimal pricing rule to be derived in Section 4.

Proposition 3 *When either R_r exceeds 1 but is falling, or when P_r and R_r exceed 3 and 1, respectively, the consumer's ex-ante demand is increasing in the expected aggregate demand X^e. The effect of a marginal increase in the expected aggregate demand on the consumer's maximal expected utility is negative under the same conditions that give $\widehat{x} > x^*$.*

Finally, we examine the effect of marginal changes in the supply distribution. For this purpose, we redefine $F(\cdot)$ as a weighted average of two probability distributions, $G(\cdot)$ and $H(\cdot)$: $F(T,\theta) \stackrel{\text{def}}{=} (1-\theta)\, G(T) + \theta H(T)$. If G first degree stochastically dominates H, then $\mathrm{d}\theta > 0$ can be thought of as a FSD-deteriorating shift and the condition is that

$$F_\theta (T, \theta) = H(T) - G(T) \geq 0; \forall\, T \in [0, \infty)\,. \tag{3.12}$$

On the other hand, if

$$\int_0^z F_\theta (T, \theta)\, \mathrm{d}T = \int_0^z \left[H(T) - G(T)\right] \mathrm{d}T \geq 0; \text{ for all } z \in [0, \infty)\,, \tag{3.13}$$

then G second order stochastically dominates H, and $\mathrm{d}\theta > 0$ can be thought of as a SSD-deteriorating shift.[7]

Using (3.4a), the effect of $\mathrm{d}\theta$ on $r_F(\cdot)$ is

$$\frac{\partial r_F(X^e)}{\partial \theta} = -\frac{1}{X_e}\int_0^{X^e} F_\theta(T,\theta)\mathrm{d}T = r_H(X^e) - r_G(X^e). \tag{3.14}$$

Thus both a first and second order dominance deteriorating shift reduce the reliability of the system.

The important observation is that the consumer's net marginal utility behaves asymmetrically around $T = X^e$ where it displays a kink.[8] This is intuitive, as in situations with $T > X^e$, no interruption occurs, and the net marginal utility is independent of the degree of excess capacity. However, with $T < X^e$, the degree of capacity shortage will affect the net marginal utility of the *ex ante* order. This is presented in Figure 1, where it is assumed that $u'\left(\frac{T}{X^e}\hat{x}\right)\frac{T}{X^e} - p\frac{T}{X^e}$ is falling in T. For this to happen, it is sufficient that $R_r(\frac{T}{X^e}\hat{x})$ exceeds 1 for all T. The net marginal utility function is now non-increasing in T, and its expected value will increase due to an FSD-deteriorating shift.[9]

For $T < X^e$, convexity of the net marginal utility is equivalent to $P_r(\frac{T}{X^e}\hat{x}) \geq 2$. Since the maximum of two convex functions is convex, the expected value of the net marginal utility function will increase due to an SSD-deteriorating shift.[10]

Formally, we may differentiate (3.5) completely to obtain:

$$\frac{\partial \hat{x}}{\partial \theta}(-SOC_{\hat{x}}) = \underbrace{-p\frac{\partial r_F(X^e)}{\partial \theta}}_{\substack{\text{marginal} \\ \text{outlay effect}}} + \underbrace{\int_0^{X^e} \widetilde{u}'\left(\widetilde{R}_r - 1\right)\frac{F_\theta(T,\theta)}{X^e}\mathrm{d}T}_{\substack{\text{marginal risk premium} \\ \text{effect}}} \tag{3.15}$$

The marginal outlay effect is clearly negative. To see the effect of an FSD-deteriorating shift, it follows immediately from (3.15) that a relative risk

[7] If in addition G and H have the same mean (i.e., $\int_0^\infty F_\theta(T,\theta)\mathrm{d}T = \int_0^\infty [H(T) - G(T)]\mathrm{d}T = 0$), then $\mathrm{d}\theta > 0$ may be considered as a mean-preserving spread (cf. Rothschild and Stiglitz, 1970), also called a 2nd degree increase in risk.

[8] The consequences kinks in the payoff function for the effects of SSD-shifts on optimal decisons were first discussed by Kanbur (1982).

[9] It can also be shown that $R_r(\frac{T}{X^e}\widehat{x}) > 1$ is a necessary condition for the expected marginal utility to increase for any arbitrary FSD-deteriorating shift.

[10] It can also be shown that $R_r(\frac{T}{X^e}\widehat{x}) > 1$ and $P_r(\frac{T}{X^e}\widehat{x}) > 2$ together are a necessary set of conditions for the expected marginal utility to increase for any arbitrary SSD-deteriorating shift.

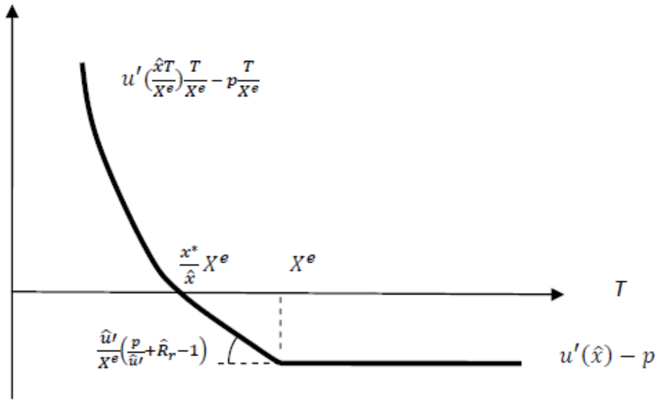

Figure 1: Net marginal utility as a function of available supply

aversion exceeding 1 is sufficient to give rise to a higher *ex ante* order. Integration by parts allows us to rewrite the marginal risk premium as

$$\widehat{u}' \left(1 - \widehat{R}_r\right) \frac{\partial r_F(X^e)}{\partial \theta} - \int_0^{X^e} \widetilde{u}' \ \widetilde{R}_r \left[\widetilde{P}_r - 2\right] \frac{\partial \widetilde{r}_F}{\partial \theta} \mathrm{d}T. \tag{3.16}$$

Thus a relative risk aversion exceeding 1 and relative prudence exceeding 2 are jointly sufficient conditions for the marginal risk premium effect to be positive for any shift in distribution that lowers reliability at all levels. Graphically, these conditions ensure that the function drawn in Figure 1 is convex and therefore that its expected value will raise above due to such shifts. The optimal response is to bring this expected value down to zero again by increasing the *ex ante* order.[11]

[11] The condition on relative prudence is reminiscent of the analysis of precautionary savings behaviour: if the rate of return to savings becomes more risky, the consumer will increase the amount saved if and only if his relative prudence exceeds 2 (this result dates back to Leland (1968); a modern account is found in Eeckhoudt and Schlesinger, 2008). Prudence needs to be high enough to place a higher order because on the one hand a more risky distribution makes the uncertain consumption of the service or good less attractive compared with the certain consumption of the *numéraire* (the substitution effect), but on the other hand, the increase in risk makes the consumer more cautious (the precautionary motive effect).

Here, we need in addition a condition on relative risk aversion. Graphically, this is easy to understand. If $\frac{p}{\widehat{u}'} + \widehat{R}_r - 1$ where negative, the net marginal benefit function would cease to be convex in the neighbourhood of the kink in Figure 1. A relative risk aversion exceeding 1 at $\hat{x}$ rules this possibility out. As we will show in the next section, stability in a rational expectations equilibrium requires precisely that $\frac{p}{\widehat{u}'} + \widehat{R}_r - 1 > 0$.

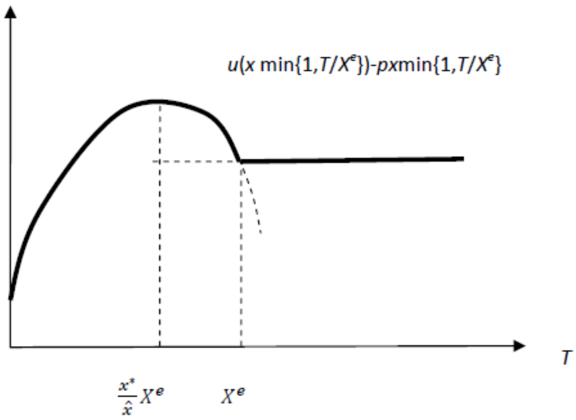

Figure 2: Utility as a function of available supply.

Proposition 4 *If $R_r > 1$, an FSD-deteriorating shift in the distribution of available capacity will result in a higher* ex ante *demand. If $R_r > 1$ and $P_r > 2$, any shift in the distribution of available capacity that reduces the reliability at all levels will result in a higher ex-ante demand.*

The effect of a perturbation of the capacity distribution function on the maximal expected utility is given by

$$\frac{\partial V}{\partial \theta} = \int_0^{X^e} \left(p - \widetilde{u}'\right) F_\theta(T)\mathrm{d}T, \tag{3.17}$$

$$= -\left(p - \widehat{u}'\right) \frac{\partial r_F(X^e)}{\partial \theta} X^e - \widehat{u}' \int_0^{X^e} \frac{\widetilde{u}'}{\widehat{u}'} \widetilde{R}_r \frac{\partial \widetilde{r}_F}{\partial \theta} \mathrm{d}T, \tag{3.18}$$

where the second equality follows after integration by parts. Inspection of either (3.17) or (3.18) shows that neither an FSD-deteriorating shift, nor an SSD-deteriorating shift need result in a fall in expected utility. The reason is that utility as a function of T, $u\left(\widehat{x}\min\{1, \frac{T}{X^e}\}\right) + m - p\widehat{x}\{1, \frac{T}{X^e}\}$, will not be monotonically increasing (FSD-deteriorating shift) nor concave (SSD-deteriorating shift) whenever $\hat{x} > x^*$. This is illustrated in Figure 2.

3.2 Comparative statics under rational expectations

Previously, we treated the anticipated aggregate demand as an exogenously defined variable. We now proceed by imposing rational expectations (RE), so that this anticipation is confirmed in equilibrium, viz., $X^e = \hat{x}\left(p, X^e, F(\cdot)\right)$.[12] This means that everywhere in the analysis, we can replace X^e by $\hat{x}$.

For expectations to be implicitly defined by the model, we need $\frac{\partial \hat{x}}{\partial X^e}|_{\hat{x}=X^e} \neq 1$. In addition, for the RE equilibrium to be stable under eductive learning, we need that $-1 < \frac{\partial \hat{x}}{\partial X^e}|_{\hat{x}=X^e} < 1$. Working out $1 - \frac{\partial \hat{x}}{\partial X^e}$ by means of (3.10) and (3.7) gives us:

$$1 - \frac{\partial \hat{x}}{\partial X^e}|_{\hat{x}=X^e} = \frac{\left(\frac{p}{\widehat{u}'} + \widehat{R}_r - 1\right)\left(1 - F(\hat{x})\right)}{\widehat{R}_r + \int_0^{\hat{x}} \frac{\widetilde{u}'}{\widehat{u}'} \widetilde{R}_r \left(\widetilde{P}_r - 2\right) \frac{F(T)}{\hat{x}} \mathrm{d}T}. \tag{3.19}$$

Assuming $\frac{\partial \hat{x}}{\partial X^e}|_{\hat{x}=X^e} > 0$, the stability condition thus amounts to:

$$\frac{p}{\widehat{u}'} + \widehat{R}_r - 1 > 0, \tag{3.20}$$

which we assume to hold from now on.

The equilibrium effects on demand from changes in output price or capacity uncertainty are then

$$\frac{\partial \hat{x}}{\partial p}|_{\mathrm{RE}} = \frac{\frac{\partial \hat{x}}{\partial p}}{1 - \frac{\partial \hat{x}}{\partial X^e}} = \frac{-\ \hat{x}\ r_F(\hat{x})}{\widehat{u}'\left[1 - \widehat{F}\right]\left[\frac{p}{\widehat{u}'} + \widehat{R}_r - 1\right]} < 0, \tag{3.21}$$

and

$$\frac{\partial \hat{x}}{\partial \theta}|_{\mathrm{RE}} = \frac{\frac{\partial \hat{x}}{\partial \theta}}{1 - \frac{\partial \hat{x}}{\partial X^e}} = \frac{-\frac{p}{\widehat{u}'}\frac{\partial \widehat{r}_F}{\partial \theta}\widehat{x} + \int_0^{\widehat{x}} \frac{\widetilde{u}'}{\widehat{u}'}\left(\widetilde{R}_r - 1\right) F_\theta\left(T\right) \mathrm{d}T}{\left[1 - \widehat{F}\right]\left[\frac{p}{\widehat{u}'} + \widehat{R}_r - 1\right]} > 0. \tag{3.22}$$

Note that the stability assumption ensures that the denominator is positive. Therefore, the same conditions that guarantee the expected sign at the individual level, will do so in equilibrium.

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[12] The simplicity with which the rational expectations equilibrium can be defined in our model is due to our assumption that consumers have identical preferences. If preferences where heterogeneous, i.e., $U(x, Y; \theta) = \theta u(x) + Y$ where θ has cdf $\Psi(\cdot)$ on support Θ, $\hat{x}\left(p, X^e, F(\cdot); \theta\right)$ solves

$$\max_x \int_0^\infty \left[\theta u\left(x \min\{1, \frac{T}{X^e}\}\right) + m - px \min\{1, \frac{T}{X^e}\}\right] \mathrm{d}F\left(T\right)$$

and the rational expectations condition becomes $X^e = \int_{\theta' \in \Theta} \hat{x}\left(p, X^e, F(\cdot); \theta'\right) \mathrm{d}\Psi(\theta')$. (This is the Bayesian Nash equilibrium concept in a game with incomplete information about types, but with common knowledge that types are drawn from the distribution $\Psi(\cdot)$ on Θ.)

We can now deduce the equilibrium effects of changes in price and supply uncertainty on consumer welfare. The former effect is

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}p}|_{RE} &= \frac{\partial V}{\partial p}|_{RE} + \frac{\partial V}{\partial X^e}|_{X^e=\hat{x}} \cdot \frac{\partial \hat{x}}{\partial p}|_{RE}, \\ &= -\widehat{x}\,\widehat{r}_F \frac{\widehat{R}_r}{\frac{p}{\widehat{u}'} + \widehat{R}_r - 1}. \end{aligned} \tag{3.23}$$

Since $\widehat{x} > x^*$, the denominator will exceed $\widehat{R}_r$. Thus the welfare effect of a price increase is less detrimental in equilibrium than at the individual level. This is because a price increase will reduce aggregate demand which in turn reduces the likelihood of being rationed off and hence improves welfare (cf (3.11)).

The effect of supply uncertainty on consumer welfare is given by

$$\frac{\mathrm{d}V}{\mathrm{d}\theta}|_{RE} = \frac{\partial V}{\partial \theta} + \frac{\partial V}{\partial X^e}|_{X^e=\hat{x}} \cdot \frac{\partial \hat{x}}{\partial \theta}|_{RE}. \tag{3.24}$$

In the appendix, we show this can be written as

$$\frac{\mathrm{d}V}{\mathrm{d}\theta}|_{RE} = \widehat{u}' \frac{\int_0^{\widehat{x}} \frac{\widetilde{u}'}{\widehat{u}'} \widetilde{R}_r \left[\widehat{R}_r - \left(1 - \frac{p}{\widehat{u}'}\right)\left(\widetilde{P}_r - 1\right)\right] \frac{\partial \widetilde{r}_F}{\partial \theta} \mathrm{d}T}{\frac{p}{\widehat{u}'} + \widehat{R}_r - 1}. \tag{3.25}$$

Recall that at the individual level, a shift in the distribution function of either an FSD or SSD type, could affect expected utility in both in a positive or negative way. Equation (3.25) shows that *any* shift that reduces reliability at all expectations levels will reduce expected consumer welfare under relative risk aversion and with relative prudence larger than 1.

We have now all the ingredients for carrying out an analysis of the optimal pricing and investment policy.

4 Welfare maximising pricing and investment

In this section, we study the optimal pricing policy, and the welfare effects of changes in the capacity distribution. For this purpose, we define social welfare as the sum of expected consumer surplus V and expected profit, while accounting for the fact that any loss that the public firm makes has to be financed through distortionary taxation on other economic activities (cf Laffont and Tirole, 1993: 24) .

Denoting the shadow cost of public funds by $\lambda > 0$, the problem of the regulator is then:

$$\max_{p \geq 0} W \stackrel{\text{def}}{=} V + (1+\lambda)(E\pi - K), \tag{4.1}$$

where V is the consumer's expected utility from (3.1) with $x = X^e$, $E\pi$ is the supplier's expected operating surplus, and K denotes fixed costs. In the appendix, we show that:

$$E\pi = (p - c)\, r_F(\widehat{x})\widehat{x}, \tag{4.2}$$

where we remind the reader that c is the marginal cost of production.[13] We then have

$$\begin{aligned} \frac{\mathrm{d}E\pi}{\mathrm{d}p}|_{RE} &= r_F(\widehat{x})\widehat{x} + (p-c)(1-F(\widehat{x}))\frac{\partial \hat{x}}{\partial p}|_{\mathrm{RE}}, \\ &= r_F(\widehat{x})\widehat{x}\frac{\frac{c}{\widehat{u}'} + \widehat{R}_r - 1}{\frac{p}{\widehat{u}'} + \widehat{R}_r - 1}, \end{aligned} \tag{4.3}$$

and also

$$\begin{aligned} \frac{\mathrm{d}E\pi}{\mathrm{d}\theta}|_{RE} &= (p-c)\left[\frac{\partial r_F(\widehat{x})}{\partial \theta}\widehat{x} + r_F(\widehat{x})\frac{\partial \hat{x}}{\partial \theta}|_{\mathrm{RE}}\right], \\ &= (p-c)\frac{\int_0^{\widehat{x}} \frac{\widetilde{u}'}{\widehat{u}'}\widetilde{R}_r\left(\widetilde{P}_r - 2\right)(\widetilde{r}_G - \widetilde{r}_H)\,\mathrm{d}T}{\frac{p}{\widehat{u}'} + \widehat{R}_r - 1}. \end{aligned} \tag{4.4}$$

An increase in θ (whether an FSD- or SSD-deterioration) has two opposite effects on expected operating surplus. On the one hand, it reduces reliability, while under the other hand it increases *ex ante* demand (under the conditions mentioned in section 3.3). Expression (4.4) shows that when consumers are sufficiently prudent, the net effect will be positive. *The manager of the utility has therefore little incentive to enhance reliability.*

4.1 The optimal pricing rule

When consumers rationally expect a reliability rate below 100%, the *ex ante* demand will satisfy (3.5) with $X^e = \hat{x}$. Using (3.23) and (4.3), the optimal

[13] Problem (4.1) reduces to profit maximisation when $\lambda \to \infty$. Alternatively, we could formulate the problem as a utility maximisation problem, subject to the constraint that the operating surplus (together with any exogenous subsidies) should cover the fixed costs. Under this alternative, $(1+\lambda)$ becomes the endogenous Lagrange multiplier to the break even constraint $E\pi \geq K$. Analytically, both approaches are equivalent.

price policy then necessarily satisfies the first-order condition:

$$\frac{\mathrm{d}W}{\mathrm{d}p}|_{RE} = -\widehat{x}\,\widehat{r}_F\,\frac{\widehat{R}_r}{\frac{p}{\widehat{u}'} + \widehat{R}_r - 1} + (1+\lambda)\;\widehat{r}_F\;\widehat{x}\,\frac{\frac{c}{\widehat{u}'} + \widehat{R}_r - 1}{\frac{p}{\widehat{u}'} + \widehat{R}_r - 1} = 0. \tag{4.5}$$

Rearranging then gives:

$$u'(\widehat{x})\left[1 - \frac{\lambda}{1+\lambda} R_r(\widehat{x})\right] = c, \tag{4.6}$$

which implicitly defines the welfare maximising level of the *ex ante* order $\widehat{x}$.

A necessary condition for a finite price $\hat{p}$ to maximise profits is that the square bracket term be positive. This imposes an upper bound on the coefficient of relative risk aversion given by $\frac{1+\lambda}{\lambda}$. If $\lambda = 0.2$ (0.3), then R_r must not exceed 6 $(4\frac{1}{3})$. The corresponding SOC is

$$u''(\widehat{x})\left[1 - \frac{\lambda}{1+\lambda}\left[P_r(\widehat{x}) - 1\right]\right] < 0, \tag{4.7}$$

requiring that $P_r(\widehat{x}) < \frac{1+2\lambda}{\lambda}$. If $\lambda = 0.2$ (0.3), then P_r must not exceed 7 $(5\frac{1}{3})$. With CRRA preferences, $P_r \equiv R_r + 1$, and both upper bound conditions are equivalent.

An important feature of (4.6) is that it is independent of the reliability rate $r_F(\widehat{x})$, and thus of the supply distribution $F(\cdot)$. Therefore, the optimal *ex ante* order under uncertainty is identical to the optimal order under adequate supply, x^*. If we denote the optimal price under supply adequacy by p^*, then (4.6) shows that this price must satisfy the familiar inverse elasticity rule

$$\frac{p^* - c}{p^*} = \frac{\lambda}{1+\lambda}\frac{1}{\varepsilon(x^*)}. \tag{4.8}$$

Earlier, we concluded that for a given price, the consumer will place a higher *ex ante* order when he expects inadequate supply, relative to when he deems supply to be adequate. It follows that the optimal price under inadequate supply, $\widehat{p}$, needs to exceed p^* to choke off the *ex ante* demand, and to equalise the demand in both cases. Intuitively, the absence of a market for contingent claims and the use of a proportional rationing rule introduces a negative externality among consumers. A single consumer neglects the fact that when placing a higher order to hedge against an uncertain delivery, he reduces the expected reliability rate of the system, thereby harming everybody else. The optimal price difference $\widehat{p} - p^*$ thus acts as Pigouvian tax to internalise this demand externality.

Replacing in the consumer's necessary condition (3.5) p by $\widehat{p}$, $\widehat{u}'$ by p^*, and rearranging, we obtain the optimal mark-up of the price under supply inadequacy relative to the optimal price under adequacy:

$$\frac{\widehat{p}-p^*}{p^*}=\frac{1}{r_F^*\ x^*}\int_0^{x^*}\left[1+\frac{\widetilde{u}'}{p^*}\left(\widetilde{R}_r-1\right)\right]F\left(T\right)\mathrm{d}T.$$

Using a first order Taylor approximation of the square bracket term around $T=x^*$, we may also write this as

$$\frac{\hat{p}-p^*}{p^*}\simeq R_r^*\left[(P_r^*-1)\frac{1-r_F^*}{r_F^*}-\frac{1}{2}(P_r^*-2)\frac{1-s_F^*}{r_F^*}\right], \tag{4.9}$$

where $s^*\stackrel{\text{def}}{=}F\left(x^*\right)E\left[(\frac{T}{x^*})^2\mid T\leq x^*\right]+[1-F(x^*)]$, i.e., the second moment of the degree of supply reliability.

This mark-up rule has a straightforward operational content, linking the size of the Pigouvian tax to the degrees of relative risk aversion and prudence, and summary statistics of system reliability. Clearly, the mark-up is increasing in risk aversion. To see the effect of prudence, note that[14]

$$\frac{\partial}{\partial P_r^*}\left(\frac{\hat{p}-p^*}{p^*}\right)\simeq R_r^*\left[\frac{1-r^*}{r^*}-\frac{1}{2}\frac{1-s^*}{r^*}\right]=\frac{R_r^*}{2r_F^*}\int_0^{x^*}\left(1-\frac{T}{x^*}\right)^2\mathrm{d}F(T)>0.$$

Intuitively, a strong degree of prudence underscores the consumer's precautionary motive when placing an order. This boosts the *ex ante* demand, and thus has to be mitigated through a higher price.

Finally, system reliability has the effect

$$\frac{\partial}{\partial r_F^*}\left(\frac{\hat{p}-p^*}{p^*}\right)\simeq-\frac{1}{2}R_r^*\left[P_r^*+(P_r^*-2)s_F^*\right],$$

which is negative when $P_r^*>2$.

Proposition 5 *The optimal ex-ante demand is independent of whether supply capacity is expected to be adequate or not. The optimal price when supply is regarded inadequate must rise above the corresponding optimal price with adequate supply in proportion to the degree of relative risk aversion, to the extent consumers are prudent, and to the extent the system is unreliable.*

In the special case of logarithmic utility, the mark-up reduces to $\frac{\widehat{p}-p^*}{p^*}=\frac{1-r^*}{r^*}$. Hence, a perceived reliability of 75% requires a price exceeding the base level by 33%.

[14]This is a *ceteris paribus* result as P_r and R_r are related through $P_r\equiv R_r+1+\frac{\mathrm{d}\log R_r(w)}{\mathrm{d}\log w}$.

4.2 Welfare effect of a reliability improving investment

Let us now look at the welfare effects of an investment that leads to a reliability improving shift in the capacity distribution. We denote this shift as $\mathrm{d}\zeta$. This amounts to a reduction in the parameter θ so that $\mathrm{d}\zeta = -\mathrm{d}\theta$. Since the optimal *ex ante* order is entirely governed by (4.6) and therefore independent of the capacity distribution $F(\cdot)$, any change in this distribution will trigger a price effect to keep the *ex ante* order at x^*.

In the appendix, it is shown that

$$\frac{\mathrm{d}E\pi}{\mathrm{d}\zeta}|_{x^*} = \frac{R_r^*}{1+\lambda}p^* x^* \frac{\partial r_F^*}{\partial \theta} + p^* \int_0^{x^*} \frac{\widetilde{u}'}{p^*}\widetilde{R}_r \left(\widetilde{P}_r - 2\right) \frac{\partial \widetilde{r}_F}{\partial \theta}\mathrm{d}T. \tag{4.10}$$

Thus, if $P_r \geq 2$, even under optimal pricing will a reliability improvement have a negative impact on the expected operating surplus of the utility.

The impact on consumer welfare is given by

$$\frac{\mathrm{d}V}{\mathrm{d}\zeta}|_{x^*} = -R_r^* p^* x^* \frac{\partial r_F^*}{\partial \theta} - p^* \int_0^{x^*} \frac{\widetilde{u}'}{p^*}\widetilde{R}_r \left(\widetilde{P}_r - 1\right) \frac{\partial \widetilde{r}_F}{\partial \theta}\mathrm{d}T, \tag{4.11}$$

and therefore positive if $P_r \geq 1$.

The effect on social welfare is then found as (4.11)+$(1+\lambda)$(4.10)$-(1+\lambda)\frac{\mathrm{d}K}{\mathrm{d}\zeta}$, where $\frac{\mathrm{d}K}{\mathrm{d}\zeta}$ is the marginal investment cost:

$$\frac{\mathrm{d}W}{\mathrm{d}\zeta}|_{x^*} = p^* \int_0^{x^*} \frac{\widetilde{u}'}{p^*}\widetilde{R}_r \left[1 - \lambda\left(\widetilde{P}_r - 2\right)\right] \frac{\partial \widetilde{r}_F}{\partial \zeta}\mathrm{d}T - (1+\lambda)\frac{\mathrm{d}K}{\mathrm{d}\zeta}. \tag{4.12}$$

The first term accounts for the welfare effects of a changes in risk exposure due to $\mathrm{d}\zeta$. The strength of this effect depends on the degree of relative risk aversion. Its sign depends on the degree of relative prudence. A sufficient condition for this welfare effect to be positive is that $P_r^* < \frac{1+2\lambda}{\lambda}$ (cf the SOC (4.7)) and that $P_r' \geq 0$.

Proposition 6 *If $P_r(T) < \frac{1+2\lambda}{\lambda}$ (all $T < x^*$), the social welfare effects of an FSD- or SSD-improving shift in the supply distribution are always positive (when ignoring the investment cost).*

Since the utility when concerned with maximising profits (or minimising losses) would never enact reliability improving investments, the above proposition underscores to need for regulation of infrastructure choices.

We conclude by investigating the investment rule (4.12) under the assumption of CRRA preferences. Then R_r and P_r are constant and related

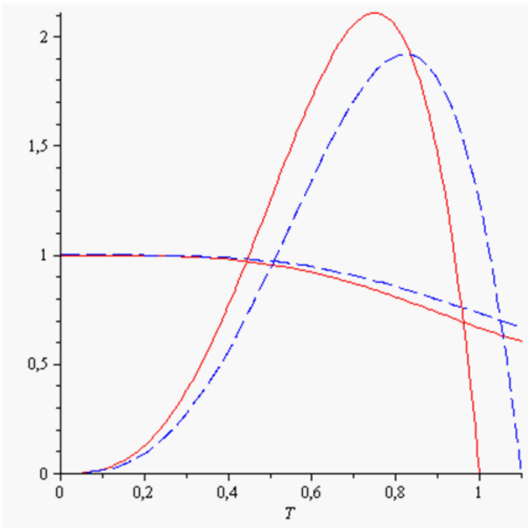

Figure 3: Density and reliability functions before (solid) and after (dashed) an FDS improvement.

as $P_r = R_r + 1$, and the square bracket term in (4.12) reduces to $(1+\lambda) - \lambda R_r$. This term will be positive iff $R_r < \frac{1+\lambda}{\lambda}$, which is exactly the earlier mentioned necessary condition for an interior optimal pricing policy. Furthermore, we can make use of the optimality condition (4.6) to rewrite this term as $(1+\lambda)\frac{c}{p^*}$. The welfare effect of a reliability improving investment then becomes

$$\frac{\mathrm{d}W}{\mathrm{d}\zeta}|_{x^*} = (1+\lambda)\left[cR_r \int_0^{x^*} \frac{\widetilde{u}'}{p^*}\frac{\partial \widetilde{r}_F}{\partial \zeta}\mathrm{d}T - \frac{\mathrm{d}K}{\mathrm{d}\zeta}\right]. \tag{4.13}$$

The appearance of the marginal cost on the benefit side is not surprising: the optimal pricing rule tells us that the marginal cost exactly measures the marginal willingness to pay, discounted for the social cost of the risk premium.

In Figure 3, the bell shaped curves depict the supply density function before (solid) and after (dashed) an FSD improving shift.[15] The monotonically downward sloping lines are the corresponding reliability functions. Expression (4.13) suggests that we should compute the benefit of this shift as the area between the reliability functions, but weighted by the ratio $\frac{u'(T)}{p^*}$.

Since $\frac{u'(T)}{p^*} = \frac{u'(T)}{u'(x^*)} > 1$ because of risk aversion, (4.13) suggests the

[15]The density function $f(T)$ corresponds to a transformed Beta distribution: $f(T) = \frac{1}{\zeta}\mathrm{Beta}(\frac{T}{\zeta}, 4, 2)$, where $\zeta = 1$ (solid) and $\zeta = 1.1$ (dashed).

following operational sufficient condition for capacity expansion:

$$\text{“expand if } cR_r \int_0^{x^*} \frac{\partial \widetilde{r}_F}{\partial \zeta} dT > \frac{\mathrm{d}K}{\mathrm{d}\zeta}.\text{”} \tag{4.14}$$

5 Conclusion

The objective of this paper has been to analyse questions of optimal pricing and capacity choice for an interruptible public service, while recognizing the interdependence of system reliability and consumer demand. Overall, the analysis shows that perceptions of system reliability play a significant role in the formation of consumer demand for a public service like electric power, water, transport, gas, etc. Furthermore, the interdependence between system reliability and demand must be taken into consideration when determining the service price and capacity investment.

In particular we have shown that when consumers are proportionally rationed in case supply falls short of aggregate demand, supply uncertainty typically leads to larger *ex ante* orders to hedge against the uncertainty. This precautionary reaction is in the same spirit of a precautionary savings increase due to increased uncertainty about the rate if return on savings. A sufficient condition for it to come about is that consumers are sufficiently risk averse and prudent.

In addition, we have shown that the welfare optimal price under supply uncertainty is one that implements the same consumption level as under certainty. This means the price must exceed the marginal willingness to pay for the socially optimal level by a mark-up that counteracts the precautionary motives for consuming more.

Finally, we have shown that even though a reliability improving investment will result in lower operating surplus, this drop will be outweighed by an increase in consumer surplus.

Our results are derived within a stylised model that could be extended in several dimensions. One is the introduction of more heterogeneity among consumers, in particular by allowing for differences in preferences for the good or service in question. We explained in footnote 12 how this should be done. A second extension is the introduction of risk aversion with respect to income. This would call for an extra instrument, to wit the use of monetary compensation in case a consumer gets rationed. In several countries, the regulatory authorities impose electricity suppliers to hand out compensations in case of interruption. One may expect that the size of the compensation (per unit undelivered) relative to the price (per unit delivered) will hinge on the difference in relative risk aversion with respect to consumption of the particular good and the numéraire.

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6 Appendix

Properties of the reliability function

$r_F(X)$ can be written as

$$\int_0^X \frac{T}{X} \mathrm{d}F(T) + [1 - F(X)].$$

Integration by parts gives

$$r_F(X) = [\frac{T}{X}F(T)]_0^X - \frac{1}{X}\int_0^X F(T)\,\mathrm{d}T + [1 - F(X)],$$

which gives (3.4a). (3.4b) follows straightforwardly by differentiating the right-hand side of (3.4a) and rearranging. To prove (3.4c), start from ((3.4a) and use the mean value theorem for integrals:

$$r_F(X) = 1 - \frac{1}{X}F(Z)X = 1 - F(Z), \text{ for some } Z \in [0, X].$$

As $X \to 0$, $Z \to 0$ even faster.

Rewriting the first order condition using integration by parts

Integration by parts the second left-hand side term in (3.2) gives:

$$\int_0^{X^e} u'\left(\frac{T}{X^e}\hat{x}\right)\frac{T}{X^e}\mathrm{d}F(T) = u'(\hat{x})F(X^e)$$
$$-\int_0^{X^e} \frac{u'\left(\frac{T}{X^e}\hat{x}\right)}{X^e}\left[\frac{u''\left(\frac{T}{X^e}\hat{x}\right)}{u'\left(\frac{T}{X^e}\hat{x}\right)}\frac{T}{X^e}\hat{x} + 1\right]F(T)\,\mathrm{d}T,$$

where it is assumed that $\lim_{T\to 0} u'\left(\frac{T}{X^e}\hat{x}\right)\frac{T}{X^e} = 0$. Then (3.2) can be written as (3.5)

The second-order condition and comparative statics.

Deriving (3.5) once again with respect to $\hat{x}$ yields the expression for the second-order condition:

$$SOC_{\hat{x}} \stackrel{\text{def}}{=} u''(\hat{x}) + \int_0^{X^e}\left\{u''\left(\frac{T}{X^e}\hat{x}\right)\frac{T}{X^e}\left[R_r\left(\frac{T}{X^e}\hat{x}\right) - 1\right]\right.$$
$$\left. + u'\left(\frac{T}{X^e}\hat{x}\right)R_r'\left(\frac{T}{X^e}\hat{x}\right)\frac{T}{X^e}\frac{F(T)}{X^e}\right\}\mathrm{d}T,$$

which we may re-express as:

$$SOC_{\hat{x}} = \widehat{u}'' + \int_0^{X^e} \widetilde{u}' \frac{T}{X^e} \left\{ -\frac{\widetilde{u}''}{\widetilde{u}'} \left[1 - \widetilde{R}_r \right] + \widetilde{R}_r' \right\} \frac{F(T)}{X^e} \mathrm{d}T. \tag{6.1}$$

Because

$$R_r' = -\frac{1}{u'}[u'' + u'''x - \frac{u''x}{u'}u''] = -\frac{u''}{u'}[1 - P_r + R_r], \tag{6.2}$$

(6.1) may be written as:

$$\begin{aligned}
SOC_{\hat{x}} &= \widehat{u}'' + \int_0^{X^e} \widetilde{u}' \frac{T}{X^e} \left\{ -\frac{\widetilde{u}''}{\widetilde{u}'} \left[1 - \widetilde{R}_r \right] + (-\frac{\widetilde{u}''}{\widetilde{u}'})[1 - \widetilde{P}_r + \widetilde{R}_r] \right\} \frac{F(T)}{X^e} \mathrm{d}T \\
&= \widehat{u}'' + \int_0^{X^e} (-\widetilde{u}'') \frac{T}{X^e} \left\{ 2 - \widetilde{P}_r \right\} \frac{F(T)}{X^e} \mathrm{d}T \\
&= -\frac{\widehat{u}'}{\widehat{x}} \left\{ \widehat{R}_r + \int_0^{X^e} \frac{\widetilde{u}'}{\widehat{u}'} \widetilde{R}_r \left(\widetilde{P}_r - 2 \right) \frac{F(T)}{X^e} \mathrm{d}T \right\},
\end{aligned}$$

which is eq (3.7) in the text.

The effect of a change in the expected aggregate demand

Because

$$\begin{aligned}
&\frac{\partial}{\partial X^e} \left(\frac{u'\left(\frac{T}{X^e}\hat{x}\right)\left[R_r\left(\frac{T}{X^e}\hat{x}\right) - 1\right]}{X^e} \right) = \\
&-\frac{\widetilde{u}'\left[\widetilde{R}_r - 1\right]}{(X^e)^2} - \frac{1}{X^e}\left(\widetilde{u}''\left[\widetilde{R}_r - 1\right] + \widetilde{u}'\widetilde{R}_r'\right)\frac{T}{(X^e)^2}\hat{x} \\
&= -\frac{\widetilde{u}'}{(X^e)^2}\left\{\left[\widetilde{R}_r - 1\right] - \widetilde{R}_r\left[\widetilde{R}_r - 1\right] + \widetilde{R}_r'\widetilde{x}\right\} \\
&= \frac{\widetilde{u}'}{(X^e)^2}\left\{\left[\widetilde{R}_r - 1\right]^2 - \widetilde{R}_r'\widetilde{x}\right\},
\end{aligned}$$

the derivative of the first-order condition with respect to X^e becomes:

$$\frac{1}{X^e}\left\{\int_0^{X^e} \widetilde{u}'\left[\left(\widetilde{R}_r - 1\right)^2 - \widetilde{R}_r'\widetilde{x}\right]\frac{F(T)}{X^e}\mathrm{d}T + \widehat{u}'\left(\widehat{R}_r - 1\right)F(X^e) - r'(X^e)X^e p\right\},$$

and therefore:

$$\begin{aligned}
\frac{\partial \hat{x}}{\partial X^e} &= -\frac{1}{X^e SOC_{\hat{x}}}\left\{\int_0^{X^e} \widetilde{u}'\left[\left(\widetilde{R}_r - 1\right)^2 - \widetilde{R}_r'\widetilde{x}\right]\frac{F(T)}{X^e}\mathrm{d}T\right. \\
&\qquad \left. + \widehat{u}'\left(\widehat{R}_r - 1\right)F(X^e) - r'(X^e)X^e p\right\}.
\end{aligned}$$

Making use of (3.7), this becomes:

$$\frac{\partial \hat{x}}{\partial X^e} = \frac{\left\{ \begin{array}{c} \int_0^{X^e} \frac{\widetilde{u}'}{\widehat{u}'} \left[\left(\widetilde{R}_r - 1 \right)^2 - \widetilde{R}_r' \widetilde{x} \right] \frac{F(T)}{X^e} \mathrm{d}T \\ + \left(\widehat{R}_r - 1 \right) F(X^e) - r'(X^e) X^e \frac{p}{\widehat{u}'} \end{array} \right\}}{\frac{X^e}{\hat{x}} \left\{ \widehat{R}_r + \int_0^{X^e} \frac{\widetilde{u}'}{\widehat{u}'} \widetilde{R}_r \left(\widetilde{P}_r - 2 \right) \frac{F(T)}{X^e} \mathrm{d}T \right\}}.$$

which is expression (3.10) in the text.

The RE stability condition

The RE stability condition is that in an equilibrium, when $X^e = \hat{x}$, $-1 < |\frac{\partial \hat{x}}{\partial X^e}|_{\mathrm{RE}} < 1$.

$$1 - \frac{\partial \hat{x}}{\partial X^e}|_{\mathrm{RE}} =$$

$$\left\{ \widehat{R}_r + \int_0^{\hat{x}} \frac{\widetilde{u}'}{\widehat{u}'} \widetilde{R}_r \left(\widetilde{P}_r - 2 \right) \frac{F(T)}{\hat{x}} \mathrm{d}T - \int_0^{\hat{x}} \frac{\widetilde{u}'}{\widehat{u}'} \left[\left(\widetilde{R}_r - 1 \right)^2 - \widetilde{R}_r' T \right] \frac{F(T)}{\hat{x}} \mathrm{d}T \right.$$

$$\left. + \left(\widehat{R}_r - 1 \right) F(\hat{x}) - r'(\hat{x}) \hat{x} \frac{p}{\widehat{u}'} \right\}$$

$$* \left\{ \widehat{R}_r + \int_0^{\hat{x}} \frac{\widetilde{u}'}{\widehat{u}'} \widetilde{R}_r \left(\widetilde{P}_r - 2 \right) \frac{F(T)}{\hat{x}} \mathrm{d}T \right\}^{-1}$$

$$= \frac{\widehat{R}_r + \int_0^{\hat{x}} \frac{\widetilde{u}'}{\widehat{u}'} \left[\widetilde{R}_r - 1 \right] \frac{F(T)}{\hat{x}} \mathrm{d}T + \left(\widehat{R}_r - 1 \right) F(\hat{x}) - r'(\hat{x}) \hat{x} \frac{p+c}{\widehat{u}'}}{\widehat{R}_r + \int_0^{\hat{x}} \frac{\widetilde{u}'}{\widehat{u}'} \widetilde{R}_r \left(\widetilde{P}_r - 2 \right) \frac{F(T)}{\hat{x}} \mathrm{d}T},$$

where use is made of the fact that $R_r' x = R_r[1 - P_r + R_r]$ (cf (6.2)). Using the first-order condition (3.5), the second term in the numerator may be replaced by $\frac{p}{\widehat{u}'} - 1 - \frac{p+c}{\widehat{u}'}(1 - r(\hat{x}))$. Keeping in mind that $-r'(\hat{x})\hat{x} = F(\hat{x}) + r(\hat{x}) - 1$, we get:

$$1 - \frac{\partial \hat{x}}{\partial X^e}|_{\mathrm{RE}} = \frac{\left(\frac{p}{\widehat{u}'} + \widehat{R}_r - 1 \right) (1 - F(\hat{x}))}{\widehat{R}_r + \int_0^{\hat{x}} \frac{\widetilde{u}'}{\widehat{u}'} \widetilde{R}_r \left(\widetilde{P}_r - 2 \right) \frac{F(T)}{\hat{x}} \mathrm{d}T}.$$

which is expression (3.19) in the text.

The utility effects in equilibrium

$$
\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}p}|_{RE} &= \frac{\partial V}{\partial p}|_{RE} + \frac{\partial V}{\partial X^e}|_{X^e=\hat{x}} \cdot \frac{\partial \hat{x}}{\partial p}|_{RE} \\
&= -\widehat{x}\ \widehat{r}_F + (\widehat{u}' - p)\,(1-\widehat{F}) \frac{-\ \hat{x}\ r_F(\hat{x})}{\widehat{u}'\left[1-\widehat{F}\right]\left[\frac{p}{\widehat{u}'} + \widehat{R}_r - 1\right]} \\
&= -\widehat{x}\ \widehat{r}_F \frac{\widehat{R}_r}{\frac{p}{\widehat{u}'} + \widehat{R}_r - 1},
\end{aligned}
$$

which is expression (3.23) in the text.

$$
\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}\theta}|_{RE} &= \frac{\partial V}{\partial \theta}|_{RE} + \frac{\partial V}{\partial X^e}|_{X^e=\hat{x}} \cdot \frac{\partial \hat{x}}{\partial \theta}|_{RE} \\
&= (p-\widehat{u}')\,[\widehat{r}_G - \widehat{r}_H]\widehat{x} - \widehat{u}' \int_0^{\widehat{x}} \frac{\widetilde{u}'}{\widehat{u}'} \widetilde{R}_r [\widetilde{r}_G - \widetilde{r}_H]\mathrm{d}T \\
&\quad + (\widehat{u}' - p)\,(1-\widehat{F})\hat{x} \frac{\left[\begin{array}{c} \left(\frac{p}{\widehat{u}'} + \widehat{R}_r - 1\right)(\widehat{r}_G - \widehat{r}_H) \\ +\frac{1}{\hat{x}} \int_0^{\widehat{x}} \frac{\widetilde{u}'}{\widehat{u}'} \widetilde{R}_r \left(\widetilde{P}_r - 2\right)(\widetilde{r}_G - \widetilde{r}_H)\,\mathrm{d}T \end{array}\right]}{\left[1-\widehat{F}\right]\left[\frac{p}{\widehat{u}'} + \widehat{R}_r - 1\right]} \\
&= \frac{\left\{\begin{array}{c} \left[\frac{p}{\widehat{u}'} + \widehat{R}_r - 1\right]\left[(p-\widehat{u}')\,[\widehat{r}_G - \widehat{r}_H]\widehat{x} - \widehat{u}' \int_0^{\widehat{x}} \frac{\widetilde{u}'}{\widehat{u}'} \widetilde{R}_r [\widetilde{r}_G - \widetilde{r}_H]\mathrm{d}T\right] \\ + (\widehat{u}' - p)\,\hat{x} \left[\begin{array}{c} \left(\frac{p}{\widehat{u}'} + \widehat{R}_r - 1\right)(\widehat{r}_G - \widehat{r}_H) \\ +\frac{1}{\hat{x}} \int_0^{\widehat{x}} \frac{\widetilde{u}'}{\widehat{u}'} \widetilde{R}_r \left(\widetilde{P}_r - 2\right)(\widetilde{r}_G - \widetilde{r}_H)\,\mathrm{d}T \end{array}\right] \end{array}\right\}}{\frac{p}{\widehat{u}'} + \widehat{R}_r - 1} \\
&= \widehat{u}' \frac{\left\{\begin{array}{c} -\widehat{R}_r \int_0^{\widehat{x}} \frac{\widetilde{u}'}{\widehat{u}'} \widetilde{R}_r [\widetilde{r}_G - \widetilde{r}_H]\mathrm{d}T \\ + \left(1 - \frac{p}{\widehat{u}'}\right)\left[\int_0^{\widehat{x}} \frac{\widetilde{u}'}{\widehat{u}'} \widetilde{R}_r \left(\widetilde{P}_r - 1\right)(\widetilde{r}_G - \widetilde{r}_H)\,\mathrm{d}T\right] \end{array}\right\}}{\frac{p}{\widehat{u}'} + \widehat{R}_r - 1} \\
&= \widehat{u}' \frac{\int_0^{\widehat{x}} \frac{\widetilde{u}'}{\widehat{u}'} \widetilde{R}_r \left[-\widehat{R}_r + \left(1 - \frac{p}{\widehat{u}'}\right)\left(\widetilde{P}_r - 1\right)\right][\widetilde{r}_G - \widetilde{r}_H]\mathrm{d}T}{\frac{p}{\widehat{u}'} + \widehat{R}_r - 1}
\end{aligned}
$$

which is expression (3.25) in the text.

The components of expected welfare and their derivatives

The expected profit for the supplier is:

$$\begin{aligned}
E\pi &= (p-c)\left[(1-F(x))x + F(x)E(T|T<x)\right] \\
&= (p-c)\,x\left[(1-F(x)) + F(x)E(\frac{T}{x}|T<x)\right] \\
&= (p-c)\,r_F(x)x, \qquad (6.3)
\end{aligned}$$

which is expression (4.2) in the text.

The derivatives of V and $E\pi$ with respect to the price are as follows.

$$V_p = [\widehat{u}'\hat{x}_p - \hat{x} - p\hat{x}_p]\,[1 - F(\hat{x})] + c\hat{x}_p F(\hat{x}) - E[T \mid T < \hat{x}]\, F(\hat{x})$$

$$\begin{aligned}
E\pi_p &= r(\hat{x})\hat{x} - (p-c)\,r_F(\hat{x})\hat{x}_p + (p-c)\,r_F'(\hat{x})\hat{x}_p \\
&= r(\hat{x})\hat{x} - [(p-c)(1-F(\hat{x}))]\,\hat{x}_p
\end{aligned}$$

where use was made of (3.4b). With $\hat{x}_p$ given by (3.21), these lead to expressions (3.23) and (4.3) in the text.

The second order condition for the optimal quantity

The first order condition was given by (4.6):

$$u'(\widehat{x})\left[1 - \frac{\lambda}{1+\lambda}R_r(\widehat{x})\right] = c.$$

Differentiating with respect to $\widehat{x}$, we get

$$\begin{aligned}
& u''(\widehat{x})\left[1 - \frac{\lambda}{1+\lambda}R_r(\widehat{x})\right] - \frac{\lambda}{1+\lambda}u'(\widehat{x})\,R_r'(\widehat{x}) \\
=\; & u''(\widehat{x})\left[1 - \frac{\lambda}{1+\lambda}R_r(\widehat{x})\right] + \frac{\lambda}{1+\lambda}\left[u'''(\widehat{x})\,\widehat{x} + u''(\widehat{x}) - \frac{u''(\widehat{x})\,\widehat{x}}{u'(\widehat{x})}u''(\widehat{x})\right] \\
=\; & u''(\widehat{x})\left\{1 - \frac{\lambda}{1+\lambda}R_r(\widehat{x}) + \frac{\lambda}{1+\lambda}\left[\frac{u'''(\widehat{x})\,\widehat{x}}{u''(\widehat{x})} + 1 - \frac{u''(\widehat{x})\,\widehat{x}}{u'(\widehat{x})}\right]\right\} \\
=\; & u''(\widehat{x})\left[1 + \frac{\lambda}{1+\lambda} - \frac{\lambda}{1+\lambda}P_r(\widehat{x})\right]
\end{aligned}$$

which gives (4.7) in the text.

The approximation for the optimal mark-up rule

The linear approximation of $u'(T)[R_r(T)-1]$ around $T=x^*$ is:

$$\begin{aligned} u'(T)[R_r(T)-1] &\simeq u'^*[R_r^*-1]+[u''^*(R_r^*-1)+u'^*R_r'^*](T-x^*) \\ &= u'^*[R_r^*-1]+u''^*[P_r^*-2](T-x^*) \\ &= u'^*\left\{R_r^*-1-R_r^*[P_r^*-2]\frac{T-x^*}{x^*}\right\}, \end{aligned}$$

where the first equality follows from (6.2).

Hence:

$$\int_0^{x^*} u'(T)[R_r(T)-1]\frac{F(T)}{x^*}\mathrm{d}T \simeq$$

$$\int_0^{x^*} u'^*\{R_r^*-1+R_r^*[P_r^*-2]\}\frac{F(T)}{x^*}\mathrm{d}T - u'^*\int_0^{x^*} R_r^*[P_r^*-2]\frac{T}{(x^*)^2}F(T)\mathrm{d}T$$

$$= u'^*[-1+R_r^*(P_r^*-1)]F^*[1-E(\frac{T}{x^*}|T<x^*)] - u'^*R_r^*[P_r^*-2]\frac{1}{2}\int_0^{x^*}\mathrm{d}(\frac{T}{x^*})^2F(T)$$

$$= u'^*\left\{[-1+R_r^*(P_r^*-1)]F^*[1-E(\frac{T}{x^*}|T<x^*)] - R_r^*[P_r^*-2]\frac{1}{2}F^*[1-E((\frac{T}{x^*})^2|T<x^*)]\right\},$$

where the second equality follows from integration by parts. Using the definition of $r(x^*)$ (cf (3.3)), we can also write this as:

$$u'^*\left\{[-1+R_r^*(P_r^*-1)](1-r^*) - R_r^*[P_r^*-2]\frac{1}{2}F^*[1-E((\frac{T}{x^*})^2|T<x^*)]\right\}.$$

This means that (3.5) can be written as:

$$r^*\hat{p}(x^*) \simeq r^*u'^*+u'^*\left\{R_r^*(P_r^*-1)(1-r^*) - R_r^*[P_r^*-2]\frac{1}{2}F^*[1-E((\frac{T}{x^*})^2|T<x^*)]\right\}.$$

Because $u'^*=p^*$, this expression may be rearranged as (4.9) in the text.

The effect of a marginal increase in θ on social welfare

$$\frac{\mathrm{d}W}{\mathrm{d}\theta} = \left(\frac{\partial V}{\partial\theta}|_{x^*}+(1+\lambda)\frac{\partial\pi}{\partial\theta}|_{x^*}\right)+\left(\frac{\partial V}{\partial p}|_{x^*}+(1+\lambda)\frac{\partial\pi}{\partial p}|_{x^*}\right)\cdot\frac{\mathrm{d}p}{\mathrm{d}\theta}|_{x^*}$$

Using (3.9), (3.18), and the fact that $\frac{\partial\pi}{\partial\theta}|_{x^*}=(p-c)x^*\frac{\partial r_F^*}{\partial\theta}$ and $\frac{\partial\pi}{\partial p}|_{x^*}=x^*r_F^*$, this may be rewritten as

$$\begin{aligned} \frac{\mathrm{d}W}{\mathrm{d}\theta} &= [\lambda p-(1+\lambda)c+u^{*\prime}]x^*\frac{\partial r_F^*}{\partial\theta}+\int_0^{x^*}\frac{\widetilde{u}'}{\widetilde{u}'}\widetilde{R}_r\frac{\partial\widetilde{r}_F}{\partial\theta}\mathrm{d}T+ \\ &\quad +\lambda x^*\left([u^{*\prime}(1-R_r^*)-p]\frac{\partial r_F^*}{\partial\theta}+\frac{u^{*\prime}}{x^*}\int_0^{x^*}\frac{\widetilde{u}'}{u^{*\prime}}\widetilde{R}_r\left(2-\widetilde{P}_r\right)\frac{\partial\widetilde{r}_F}{\partial\theta}\mathrm{d}T\right) \end{aligned}$$

Rearranging, multiplying through by -1 to get the effect of $\mathrm{d}\zeta$, and subtracting $(1+\lambda)\frac{\mathrm{d}K}{\mathrm{d}\zeta}$ then produces (4.12) in the text.

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